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Hamiltonian structure of discrete soliton systems

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Abstract
We describe an approach for investigating the Hamiltonian structures of the lattice isospectral evolution equations associated with a general discrete spectral problem. By using the so-called implicit representations of the isospectral flows, we demonstrate the existence of the recursion operator L, which is a strong and hereditary symmetry of the flows. It is then proven that every equation in the isospectral hierarchy possesses the Hamiltonian structure if L has a skew-symmetric factorization and the first equation ($u_t = K^{(0)}$) in the hierarchy satisfies some simple condition. We obtain related properties, such as the implectic-symplectic factorization of L, the Liouville complete integrability and the multi-Hamiltonian structures of the isospectral hierarchy. Four examples are given.

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1. Introduction

It is well known that an integrable soliton system possesses a remarkably rich algebraic character, i.e. infinitely many symmetries and conserved quantities, the existence of a multi-Hamiltonian formulation, etc [1]. In terms of the Hamiltonian structure, in 1986 Tu [2] proposed a successful method (the developed version has been given in [3] and [4]) for finding isospectral evolution equations and their Hamiltonian structures from the eigenvalue problem. The essence of Tu's method was a trace identity derived by the use of the chain rule of variational derivatives. Using this trace identity, the functional gradient can be obtained simply [5–8]. Another classical approach to constructing the Hamiltonian structures of integrable systems has been developed by Fokas and co-workers [1, 9, 10]. In this method, a certain operator L, called the recursion operator, plays a central role. Actually, the operator should first be a strong and hereditary symmetry for the isospectral evolution equations derived from the eigenvalue problem, and, further, it can be factorized in terms of the two Hamiltonian operators. Then the Hamiltonian structure can be established by using the properties of the operator and evolution equations. However, not all the recursion operators are hereditary [11].

So, we need a simple approach, especially for a discrete system, to obtain this important property of the recursion operator, but without being involved in a long and tedious verification as previously [12, 13].

In this paper, we investigate a general discrete hierarchy by using the implicit representation theory, proposed by Chen and Zhang [14, 15], by which the evolution equations, the symmetries of isospectral equations and their algebraic structures can be constructed from the concerned eigenvalue problem in a simple way. First, in the light of the theory, we obtain the evolution equation hierarchy and the related recursion operator L, which is a strong and hereditary symmetry of the whole hierarchy, from the following general discrete linear problem pair

$$E\phi = M\phi = M(\lambda, u(t,n))\phi \qquad \phi_t = N\phi = N(\lambda, u(t,n))\phi \tag{1}$$

where E is a shift operator defined in section 2. Then, on the basis of the results of Fokas and co-workers [1, 9, 10] we prove that every equation in the hierarchy has the Hamiltonian structure if L possesses a skew-symmetric factorization and the first equation ($u_t = K^{(0)}$) in the hierarchy satisfies some simple condition. In addition, we obtain some related properties, such as the implectic-symplectic factorization of L, the Liouville integrability and the multi-Hamiltonian structures of the evolution equations. As applications, we discuss the Toda lattice, the three-field Blaszak–Marciniak (B–M) lattice, the Ablowitz–Ladik lattice and a new lattice (which we call the B–M(II) lattice).

Although the object which we investigate is a general discrete system, the method in this paper can obviously be applied to the continuous case as well.

The paper is organized as follows. In section 2 we recollect some basic notions for discrete soliton systems. In section 3, using the implicit representation theory, we obtain the isospectral evolution equation hierarchy, the related recursion operator L and the property of L, which is a strong and hereditary symmetry. In section 4 the Hamiltonian structures for discrete evolution equations are established and the related properties are given. Finally, in section 5 we give four examples.

2. Basic notions

We assume that $U_s = \{u(t,n) = (u_1, u_2, \ldots, u_s)^T\}$ is an s-dimensional vector field space, where $u_i = u_i(t,n)$, $1 \leqslant i \leqslant s$ are all real functions defined over $R \times Z$, and vanish rapidly as $|n| \to \infty$. Let $\mathcal{V}_l$ denote a linear space containing all vector fields $f = (f_1, f_2, \ldots, f_l)^T$ living on U_s. Here $f_i = f_i(u(t,n))$, $1 \leqslant i \leqslant l$, are C^∞ differentiable with respect to t and n, C^∞ Gateaux differentiable with respect to u, and $f_i|_{u=0} = 0$. Then, let $\mathcal{Q}_m(\lambda)$ denote a Laurent matrix polynomial space composed by all $m \times m$ matrices $Q = Q(\lambda, u(t,n)) = (q_{ij}(\lambda, u(t,n)))_{m\times m}$, where q_{ij} (or Q) are all the Laurent (matrix) polynomials of λ. We also introduce two subspaces $\mathcal{Q}_m^{(l+)}(\lambda)$ and $\mathcal{Q}_m^{(l-)}(\lambda)$ described respectively by

$$\mathcal{Q}_m^{(l+)}(\lambda) = \{Q \in \mathcal{Q}_m(\lambda) \mid \text{the lowest degree of } Q \geqslant l\}$$

and

$$\mathcal{Q}_m^{(l-)}(\lambda) = \{Q \in \mathcal{Q}_m(\lambda) \mid \text{the highest degree of } Q \leqslant l\}.$$

Now we define the shift operator E as

$$Ef(n) = f(n+1) \qquad E^{-1}f(n) = f(n-1) \qquad n \in Z$$

and the difference operator $\Delta = E - E^{-1}$. Sometimes, without confusion and for convenience, we write $f(n) = f = f_n, E^k f(n) = f(n+k) = f_{n+k}, k \in Z$. We also define the inverse operator of Δ as

$$\Delta^{-1} f_n = -\sum_{k=0}^{\infty} f_{n+2k+1} \qquad \text{or} \qquad \Delta^{-1} f_n = \sum_{k=-\infty}^{0} f_{n+2k-1}$$

where we please f_n sufficiently near to zero as $|n| \to \infty$.

In what follows, we recall many necessary definitions and basic notions for the discrete systems.

Definition 1. *The Gateaux derivative of $f \in \mathcal{V}_s$ (f is an operator on $\mathcal{V}_s$) in the direction $g \in \mathcal{V}_s$ is defined by*

$$f'[g] = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} f(u + \varepsilon g). \tag{2}$$

For example, if $f = f(u(t, n)) \in \mathcal{V}_s$, then

$$f' = \sum_j \frac{\partial f}{\partial(E^j u)} E^j.$$

Using the Gateaux derivative, the Lie product for any $f, g \in \mathcal{V}_s$ can be described as

$$[\![f, g]\!] = f'[g] - g'[f]. \tag{3}$$

Definition 2. *For a given discrete evolution equation*

$$u_t = K(u(t, n)), \tag{4}$$

$\sigma(u(t, n)) \in \mathcal{V}_s$ is called its symmetry if $\sigma_t = K'[\sigma]$. All the symmetries of equation (4) form a linear space S, whose adjoint $S^* = \{\gamma(u(t, n)) \mid -\gamma_t = K'^* \gamma\}$ denotes a conserved covariant space.

We note that, throughout this paper, we always assume that the symmetries, conserved covariants and operators which we investigate do not contain t explicitly.

Definition 3. *Suppose that L and Ω are operators on $\mathcal{V}_s$. L is called the strong symmetry of equation (4) if $L : S \mapsto S^*$, i.e.*

$$L'[K] - [K', L] = 0. \tag{5}$$

If L satisfies

$$L'[Lf]g - L'[Lg]f = L(L'[f]g - L'[g]f) \qquad \forall f, g \in \mathcal{V}_s \tag{6}$$

then L is called hereditary or a hereditary symmetry [16], meaning that if L is a strong symmetry for $u_t = K$, L is also a strong symmetry for $u_t = LK$, as for $u_t = L^l K$, $(l = 2, 3, \ldots)$. Furthermore, it is obvious that if L is a strong symmetry for equation (4), then there must exist $L^* : S^* \mapsto S^*$. Also, if L is hereditary for symmetry, then L^* is also hereditary for the conserved covariant. The operator Ω is called a Noether operator for equation (4), if $\Omega : S^* \mapsto S$, i.e.

$$\Omega'[K] = \Omega K'^* + K'\Omega. \tag{7}$$

Definition 4. *Let $J(u)$ and $\theta(u)$ be two skew-symmetric operators on $\mathcal{V}_s$. Then $J(u)$ is symplectic if*

$$(f, J'[g]h) + (g, J'[h]f) + (h, J'[f]g) = 0 \qquad \forall f, g, h \in \mathcal{V}_s \tag{8}$$

and $\theta(u)$ is inverse-symplectic (implectic for short [19]) if

$$(f, \theta'[\theta g]h) + (g, \theta'[\theta h]f) + (h, \theta'[\theta f]g) = 0 \qquad \forall f, g, h \in \mathcal{V}_s. \tag{9}$$

Here, the inner product $(\cdot, \cdot)$ is defined by $(f, g) = \sum_{n=-\infty}^{\infty} f_n^T g_n$, $f, g \in \mathcal{V}_s$.

Definition 5. *Consider a real-value functional $H = H(u(t, n))$ defined over U_s. If, for every $h(u(t, n)) \in \mathcal{V}_s$, there is $f(u(t, n)) \in \mathcal{V}_s$, such that $H'[h] = (f, h)$, then we say that $f(u(t, n))$ is a functional derivative or gradient of $H(u(t, n))$, denoted by*

$$f(u(t, n)) = \frac{\delta H}{\delta u} = \nabla H = \operatorname{grad} H.$$

Hence, the Poisson bracket of two functionals $W = W(u(t, n))$ and $H = H(u(t, n))$, both defined over U_s, is described as

$$\langle W, H\rangle = \left(\frac{\delta W}{\delta u}, \theta \frac{\delta H}{\delta u}\right)$$

in which θ is an implectic operator because of the Jacobi identity of $\langle\cdot, \cdot\rangle$.

Finally, in this section, we describe the Hamiltonian structure of the discrete nonlinear evolution equation (4).

Definition 6. *Equation (4) is said to be a Hamiltonian system if it can be written in the form*

$$u_t = \theta(u) f(u) = \theta(u) \frac{\delta H}{\delta u} \tag{10}$$

where θ is implectic and f is the gradient of H.

3. Implicit representations of isospectral flows $\{K^{(l)}\}$ and property of recursion operator L

In this section, we introduce the implicit representations [14, 15] of isospectral flows $\{K^{(l)}\}$. We construct the recursion operator L and we prove that L is a strong and hereditary symmetry of the flows.

Consider the general discrete linear problem (1). The corresponding integrability condition, the discrete zero curvature equation, is

$$M_t = (EN)M - MN. \tag{11}$$

Generally, from this, the discrete evolution equation hierarchy can be obtained

$$u_{nt} = K^{(l)}(u(t, n)) = L^l K^{(0)}(u(t, n)) \qquad l = 0, 1, \ldots \tag{12}$$

where L is a recursion operator and $\{K^{(l)}\}$ are isospectral flows. Of course, the spectral matrix M is the same for all flows of the hierarchy whereas the time matrix N differs from flow to flow. Here and below, we let $N^{(l)}$ denote N which leads to the flow $K^{(l)}$. Then, if we note that $M_t = M'[u_{nt}]$, we can further find from equation (11) that

$$M'[K^{(l)}] = (EN^{(l)})M - MN^{(l)} \qquad l = 0, 1, 2, \ldots. \tag{13}$$

These equations are called the implicit representations of isospectral equations (12) (or the flows $\{K^{(l)}\}$).

Theorem 1. *Suppose that the linear problem (1) satisfies the following conditions:*

(1) the matrix equation

$$M'[X] = (EN)M - MN \tag{14}$$

possesses a unique couple of non-zero solutions $X(u(t,n)) \in \mathcal{V}_s$ *and* $N(\lambda, u(t,n)) \in \mathcal{Q}_m(\lambda)$ *satisfying* $N|_{u=0} = N_0$, *where* N_0 *is a matrix independent of* u *and meeting* $M|_{u=0}N_0 = N_0 M|_{u=0}$;

(2) for any given $Y(u(t,n)) \neq 0 \in \mathcal{V}_s$, *there exist solutions* $X(u(t,n)) \in \mathcal{V}_s$ *and* $N(\lambda, u(t,n)) \in \mathcal{Q}_m(\lambda)$ *satisfying*

$$M'[X - \lambda^\alpha Y] = (EN)M - MN \qquad N|_{u=0} = 0 \tag{15}$$

where α *is a constant related to problem (1). Then the following results are right:*

(1) there must exist the implicit representations (13) for isospectral flows $\{K^{(l)}\}$;

(2) there exists the unique recursion operator L *such that*

$$u_{n\,t} = K^{(l)}(u(t,n)) = L^l K^{(0)}(u(t,n)) \qquad l = 0, 1, \ldots \tag{16}$$

and L *must be a strong and hereditary symmetry of equation (16).*

Proof. The first flow $K^{(0)}(u(t,n))$ can be found as a solution of equation (14), i.e.

$$M'[K^{(0)}] = (EN^{(0)})M - MN^{(0)} \qquad K^{(0)} \in \mathcal{V}_s \qquad N^{(0)} \in \mathcal{Q}_m(\lambda). \tag{17}$$

Then the second flow $K^{(1)}(u(t,n))$ appears from

$$M'[K^{(1)} - \lambda^\alpha K^{(0)}] = (EU^{(1)})M - MU^{(1)} \qquad U^{(1)}|_{u=0} = 0 \in \mathcal{Q}_m(\lambda) \tag{18}$$

and so do other flows $\{K^{(l)}\}$, $(l = 2, 3, \ldots)$. Then, it can easily be shown that

$$M'[K^{(l)}] = (EN^{(l)})M - MN^{(l)} \qquad N^{(l)} = \sum_{j=0}^{l} U^{(j)}\lambda^{\alpha(l-j)} \in \mathcal{Q}_m(\lambda) \qquad U^{(0)} = N^{(0)}. \tag{19}$$

The conditions of the theorem suggest that for any $Y \neq 0 \in \mathcal{V}_s$ there exists a unique mapping $L : \mathcal{V}_s \longmapsto \mathcal{V}_s$ such that $X = LY$. Obviously, L is just the recursion operator for the evolution equation hierarchy $\{K^{(l)}\}$ related to the linear problem pair (1). In addition, we can also find that for any $Y \neq 0 \in \mathcal{V}_s$ there exists $\overline{N}^{(k)}(\lambda, u(t,n)) \in \mathcal{Q}_m(\lambda)$ meeting

$$M'[L^k Y - \lambda^{\alpha k} Y] = (E\overline{N}^{(k)})M - M\overline{N}^{(k)} \qquad \overline{N}^{(k)}|_{u=0} = 0 \qquad (k = 1, 2, \ldots). \tag{20}$$

Next, with the help of the equality

$$L'[[\![f, g]\!]] = (L'[f])'[g] - (L'[g])'[f]$$

equation (17) coupled with equation (20) (taking $k = 1$) yields

$$M'[[\![LY - \lambda^\alpha Y, K^{(0)}]\!]] = (E\widetilde{N})M - M\widetilde{N} \qquad \widetilde{N}|_{u=0} = 0 \tag{21}$$

where

$$\widetilde{N} = \overline{N}^{(1)'}[K^{(0)}] - N^{(0)'}[LY - \lambda^\alpha Y] + [\overline{N}^{(1)}, N^{(0)}].$$

Then, noticing that

$$[\![LY - \lambda^\alpha Y, K^{(0)}]\!] = (L'[K^{(0)}] - [K^{(0)'}, L])Y + L[\![Y, K^{(0)}]\!] - \lambda^\alpha [\![Y, K^{(0)}]\!]$$

we can find that

$$M'[(L'[K^{(0)}] - [K^{(0)'}, L])Y] = (EU)M - MU \qquad U|_{u=0} = 0 \tag{22}$$

where $U = \widetilde{N} - \widehat{N}$, and $\widehat{N}$ is a solution of equation (20) (taking $k = 1$) as Y is replaced by $[\![Y, K^{(0)}]\!]$. The above equation suggests

$$(L'[K^{(0)}] - [K^{(0)'}, L])Y = 0 \tag{23}$$

which means L is a strong symmetry of the equation $u_{nt} = K^{(0)}$.

In the following, for any $Y, Z \in \mathcal{V}_s$ we have

$$M'[L'[LZ]Y - L'[LY]Z - L(L'[Z]Y - L'[Y]Z)] = (EQ)M - MQ \qquad Q|_{u=0} = 0$$
$$Q = \overline{N}^{(1)'}[LZ - \lambda^\alpha Z] - R'[LY - \lambda^\alpha Y] + [\overline{N}^{(1)}, R] - V - W$$

where R and V are the solutions of equation (20) (taking $k = 1$) as Y is replaced by Z and $[\![Y, LZ]\!] - [\![Z, LY]\!]$ respectively, and W is a solution of equation (20) when $k = 2$ and Y is substituted by $[\![Z, Y]\!]$. It then turns out that

$$L'[LZ]Y - L'[LY]Z - L(L'[Z]Y - L'[Y]Z) = 0 \qquad \forall Y, Z \in \mathcal{V}_s \tag{24}$$

namely, L is a hereditary symmetry. Thus we complete the proof. □

We note that the theorem is also right if we replace $\mathcal{Q}_m(\lambda)$ by its subspace, and the condition (1) of this theorem can be replaced equivalently by the following. Equation (14) possesses a couple of non-zero solutions and further only admits zero solutions in $\mathcal{V}_s$ and $\mathcal{Q}_m(\lambda)$ as $N|_{u=0} = 0$.

Corollary. *All the flows* $\{K^{(k)}\}$ *are symmetries of every equation in the hierarchy (16) and satisfying*

$$[\![K^{(l)}, K^{(k)}]\!] = 0 \qquad l, k = 0, 1, \ldots. \tag{25}$$

4. Hamiltonian structure and Liouville integrability

In this section we wish to investigate the Hamiltonian properties of the whole discrete hierarchy (16).

Theorem 2. *Suppose that L is a strong and hereditary symmetry for every equation in the hierarchy (16). There are two skew-symmetric operators* $\theta(u)$ *and* $J(u)$ *on* $\mathcal{V}_s$ *such that*

$$L = \theta J. \tag{26}$$

The first equation $u_{nt} = K^{(0)}$ *in equation (16) can be written in the form*

$$u_{nt} = \theta f^{(0)}(u(t,n)) = \theta \frac{\delta H^{(0)}}{\delta u} \qquad H^{(0)} = \int_0^1 (f^{(0)}(\rho u), u)\, \mathrm{d}\rho. \tag{27}$$

Then every equation in (16) is a Hamiltonian system

$$u_{nt} = K^{(l)}(u(t,n)) = \theta f^{(l)}(u(t,n)) = \theta \frac{\delta H^{(l)}}{\delta u} = \theta L^{*l} f^{(0)}(u(t,n)) \qquad l = 0, 1, \ldots \tag{28}$$

where

$$H^{(l)} = \int_0^1 (f^{(l)}(\rho u), u)\, \mathrm{d}\rho. \tag{29}$$

Now, let us prove this theorem through the following six lemmas.

Lemma 1. *$f(u(t,n)) \in \mathcal{V}_s$ is a gradient of some real-value functional H if and only if $f' = f'^*$ [9]. Here the potential H is given by*

$$H = \int_0^1 (f(\rho u), u)\, \mathrm{d}\rho. \tag{30}$$

Lemma 2. *If $f = \delta H/\delta u$ and H is a conserved quantity of equation (4) then f is a conserved covariant of equation (4) [9].*

Lemma 3. *Under the conditions of theorem 2, $H^{(0)}$ in equation (27) is a conserved quantity of the whole hierarchy (16).*

Proof. The equation $u_{nt} = K^{(l)}$ in the hierarchy (16) can be written as

$$u_{nt} = \theta L^{*l} f^{(0)}(u(t,n)) = \theta f^{(l)}(u(t,n)) \qquad (L^* = J\theta). \tag{31}$$

Then the lemma holds because of

$$\begin{aligned}\frac{\mathrm{d}H^{(0)}}{\mathrm{d}t} &= H^{(0)\prime}[u_{nt}] = (f^{(0)}, K^{(l)}) = (f^{(0)}, \theta L^{*l} f^{(0)}) = -(\theta f^{(0)}, L^{*l} f^{(0)}) \\ &= -(L^l \theta f^{(0)}, f^{(0)}) = -(L^l K^{(0)}, f^{(0)}) = -(K^{(l)}, f^{(0)}) = 0.\end{aligned}$$

□

Lemma 4. *If L is a strong and hereditary symmetry for equation (16), $f = \delta H/\delta u$ and H is a conserved quantity of the whole hierarchy (16), then $f^{(m)} = L^{*m} f$, $(m = 0, 1, \ldots)$ are all gradients [10].*

Lemma 5. *Under the conditions of theorem 2, θ is a Noether operator for every equation in the hierarchy (16).*

Proof. It is not difficult to find that $K^{(m)}$ and $L^{*m} f^{(0)}$ are respectively the symmetry and conserved covariant of every equation in (16). So, because of

$$K^{(m)} = \theta L^{*m} f^{(0)} \qquad m = 0, 1, \ldots$$

it appears reasonable to conclude that θ is a Noether operator for the whole hierarchy (16).

□

Lemma 6. *[9] Under the conditions of theorem 2, if θ is a Noether operator for the whole hierarchy (16), and all $\{f^{(m)}\}$ are gradients, then θ is implectic.*

The above six lemmas suggest that theorem 2 holds.

Theorem 3. *Under the assumptions of theorem 2, the equations in the hierarchy (16) are all integrable in the Liouville sense.*

Proof. The skew-symmetric factorization (26) implies $L\theta = \theta L^*$, by which it can be verified that

$$\begin{aligned}\langle H^{(m)}, H^{(n)}\rangle &= (f^{(m)}, \theta f^{(n)}) = (L^{*m} f^{(0)}, \theta L^{*n} f^{(0)}) = (f^{(0)}, L^m \theta L^{*n} f^{(0)}) \\ &= (f^{(0)}, \theta L^{*m+n} f^{(0)}) = -(\theta L^{*m+n} f^{(0)}, f^{(0)}) = 0 \qquad m, n = 0, 1, 2, \ldots\end{aligned}$$

namely, $H^{(m)}$ and $H^{(n)}$ are involutive. With this in mind, for any equation (31) in (16), we have

$$\begin{aligned}\frac{\mathrm{d}H^{(m)}}{\mathrm{d}t} &= H^{(m)\prime}[u_{nt}] = \left(\frac{\delta H^{(m)}}{\delta u}, K^{(l)}\right) = \left(\frac{\delta H^{(m)}}{\delta u}, \theta \frac{\delta H^{(l)}}{\delta u}\right) \\ &= \langle H^{(m)}, H^{(l)}\rangle = 0 \qquad m = 0, 1, 2, \ldots.\end{aligned}$$

Now the theorem holds. □

Theorem 4. *Under the assumptions of theorem 2, equation (31) in the hierarchy (16) possesses multi-Hamiltonian [9, 17] structures:*

$$u_{nt} = \theta\frac{\delta H^{(l)}}{\delta u} = \theta L^*\frac{\delta H^{(l-1)}}{\delta u} = \cdots = \theta L^{*k}\frac{\delta H^{(l-k)}}{\delta u} = \cdots = \theta L^{*l}\frac{\delta H^{(0)}}{\delta u} \qquad (0 \leqslant k \leqslant l). \tag{32}$$

Proof. We only need to show that θL^{*k} is implectic. In fact, θ is a Noether operator of every member in equation (16). Meanwhile $L^* : S^* \mapsto S^*$, so θL^{*k} is also a Noether operator for the whole hierarchy (16), and furthermore it is implectic in the light of lemma 6. □

Theorem 5. *Under the assumptions of theorem 2, J must be a symplectic operator, i.e. equation (26) is an implectic-symplectic factorization.*

Proof. Theorem 2 shows that θ is a Noether operator for the hierarchy (16), so it is easy to see that J is an inverse Noether operator of equation (31), which suggests

$$(\theta h, (J'[K^{(l)}] + K^{(l)'^*}J + JK^{(l)'})\theta g) = 0 \qquad \forall g, h \in \mathcal{V}_s.$$

Then, noticing that θ is implectic, one can find that

$$\begin{aligned}(g, (\theta'[LK^{(l)}] - \theta(LK^{(l)})'^* - (LK^{(l)})'\theta)h) &= (K^{(l)}, J'[\theta g]\theta h)\\ &\quad + (\theta g, J'[\theta h]K^{(l)}) + (\theta h, J'[K^{(l)}]\theta g) \qquad l = 0, 1, \ldots.\end{aligned}$$

This completes the proof. □

5. Applications

In this section, we study four discrete soliton systems and construct their Hamiltonian structures. These four examples, as the representatives of various lattice hierarchies, show that the conditions of theorem 1, so easy, can be met naturally in the process of deriving isospectral hierarchy from the concerned zero curvature equation. The first example is the Toda lattice [18] which often serves as a useful guide in studies on nonlinear waves.

5.1. Toda Lattice

In this case, the linear problem pair (1) is [19–21]

$$\begin{aligned} &E\phi = M\phi \qquad M = \begin{pmatrix} 0 & 1 \\ -p_n & \lambda - v_n \end{pmatrix} \qquad u_n = \begin{pmatrix} \ln p_n \\ v_n \end{pmatrix} \qquad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \\ &\phi_t = N\phi \qquad N = \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix}. \end{aligned} \tag{33}$$

Now we consider the matrix equation

$$M'[X - \lambda Y] = (EN)M - MN \tag{34}$$

where $X = X(u(t, n)) = (X_1, X_2)^T$ and $Y = Y(u(t, n)) = (Y_1, Y_2)^T$. This suggests that

$$A_n = -q_n B_{n+1} \qquad C_n = -(\lambda - v_{n-1})B_n + D_{n-1} \tag{35}$$

and

$$X = \lambda Y + (L_1 - \lambda L_2)\begin{pmatrix} D_n \\ B_n \end{pmatrix} \tag{36}$$

in which

$$L_1 = \begin{pmatrix} \Delta & (E-1)v_{n-1} \\ v_n(E-1) & Eq_nE - q_n \end{pmatrix} \qquad L_2 = \begin{pmatrix} 0 & E-1 \\ E-1 & 0 \end{pmatrix}. \tag{37}$$

In the case of $Y = 0$, taking $(D_n, B_n)^T = (0, 1)^T$, we can find the non-zero solutions

$$X = K^{(0)}(u(t,n)) = \begin{pmatrix} v_n - v_{n-1} \\ p_{n+1} - p_n \end{pmatrix} \in \mathcal{V}_2 \tag{38}$$

and

$$N = N^{(0)}(\lambda, u(t,n)) = \begin{pmatrix} -\lambda + v_{n-1} & 1 \\ -p_n & 0 \end{pmatrix} \in \mathcal{Q}_2^{(0+)}(\lambda) \tag{39}$$

satisfying the matrix equation

$$M'[X] = (EN)M - MN. \tag{40}$$

On the other hand, we consider the case of $Y \neq 0$. Setting $(D_n, B_n)^T$ to be independent of λ in equation (36), we have

$$\begin{pmatrix} D_n \\ B_n \end{pmatrix} = L_2^{-1}Y \qquad X = LY \tag{41}$$

where L is just the recursion operator described by

$$L = L_1 L_2^{-1} = \begin{pmatrix} (E-1)v_{n-1}(E-1)^{-1} & 1 + E^{-1} \\ (Ep_nE - p_n)(E-1)^{-1} & v_n \end{pmatrix}. \tag{42}$$

Then it is easy to find the solutions $X \in \mathcal{V}_2$ and $N \in \mathcal{Q}_2^{(0+)}(\lambda)$ satisfying $N|_{u=0} = 0$. In addition, if we set $Y = 0$ in equation (36) and expand $(D_n, B_n)^T$ in the following form

$$\begin{pmatrix} D_n \\ B_n \end{pmatrix} = \sum_{j=0}^{k} \begin{pmatrix} d_n^{(j)} \\ b_n^{(j)} \end{pmatrix} \lambda^{k-j}$$

it is not difficult to find that equation (40) only admits zero solutions in $\mathcal{V}_2$ and $\mathcal{Q}_2^{(0+)}(\lambda)$ if $N|_{u=0} = 0$. Thus according to theorem 1, the isospectral hierarchy of the Toda lattice is

$$u_{nt} = K^{(l)}(u(t,n)) = L^l K^{(0)}(u(t,n)) \qquad l = 0, 1, \ldots \tag{43}$$

and the recursion operator L is a strong and hereditary symmetry for the whole hierarchy (43).

Next, we factorize the recursion operator L into

$$L = \theta J \tag{44}$$

where

$$\theta = \begin{pmatrix} 0 & 1 - E^{-1} \\ E - 1 & 0 \end{pmatrix} \tag{45}$$

and

$$J = \begin{pmatrix} (E-1)^{-1}(Ep_n - p_nE^{-1})(1-E^{-1})^{-1} & (E-1)^{-1}v_n \\ -v_n(E^{-1}-1)^{-1} & (E-1)^{-1}(E+1) \end{pmatrix} \tag{46}$$

are both skew-symmetric. This factorization allows us to rewrite the hierarchy (43) in the following form:

$$u_{nt} = \theta f^{(l)}(u(t,n)) = \theta L^{*l} f^{(0)}(u(t,n)) \qquad f^{(0)} = (p_n, v_n)^T \qquad l = 0, 1, \ldots. \tag{47}$$

It is easy to check

$$f^{(0)\prime} = f^{(0)\prime *} = \begin{pmatrix} p_n & 0 \\ 0 & 1 \end{pmatrix}$$

which means the first equation $u_{nt} = K^{(0)}$ can be written as

$$u_{nt} = \theta f^{(0)}(u(t,n)) = \theta \frac{\delta H^{(0)}}{\delta u} \qquad H^{(0)} = \int_0^1 (f^{(0)}(\rho u), u)\, \mathrm{d}\rho.$$

So, according to the results in section 4, equation (44) is an implectic-symplectic factorization, the hierarchy (43) is a complete Liouville integrable and the Hamiltonian structures are

$$u_{nt} = \theta \frac{\delta H^{(l)}}{\delta u} = \theta L^* \frac{\delta H^{(l-1)}}{\delta u} = \cdots = \theta L^{*l} \frac{\delta H^{(0)}}{\delta u} = \theta L^{*l} f^{(0)}(u(t,n)) \qquad l = 0, 1, \ldots \tag{48}$$

where the conserved quantities $H^{(l)}$ are obtained from equation (29). The first two conserved quantities $H^{(0)}(u(t,n))$ and $H^{(1)}(u(t,n))$ are respectively

$$H^{(0)}(u(t,n)) = \int_0^1 (f^{(0)}(\rho u), u)\, \mathrm{d}\rho = \int_0^1 \left(\begin{pmatrix} \mathrm{e}^{\rho \ln p_n} \\ \rho v_n \end{pmatrix}, \begin{pmatrix} \ln p_n \\ v_n \end{pmatrix} \right) \mathrm{d}\rho$$
$$= \sum\nolimits_{n=-\infty}^{+\infty} \left(p_n + \frac{1}{2} v_n^2 - 1 \right)$$

and

$$H^{(1)}(u(t,n)) = \int_0^1 (f^{(1)}(\rho u), u)\, \mathrm{d}\rho = \sum_{n=-\infty}^{+\infty} \left[p_n(v_n + v_{n-1}) + \frac{1}{3} v_n^3 - 1 \right].$$

5.2. Blaszak–Marciniak lattice

Our second example is the three-field Blaszak–Marciniak (B–M) lattice [6, 20, 21] or the sub-KP lattice [21], whose Lax pair is

$$E\phi = M\phi \qquad M = \begin{pmatrix} 0 & 1 & 0 \\ p_n - \lambda & q_n & 1 \\ r_n & 0 & 0 \end{pmatrix} \qquad u_n = \begin{pmatrix} q_n \\ \ln r_n \\ p_n \end{pmatrix} \qquad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$$
$$\phi_t = N\phi \qquad N = \begin{pmatrix} A_n & B_n & C_n \\ D_n & E_n & F_n \\ G_n & H_n & I_n \end{pmatrix}. \tag{49}$$

From the matrix equation

$$M'[X - \lambda Y] = (EN)M - MN \tag{50}$$

where $X = X(u(t,n)) = (X_1, X_2, X_3)^T$ and $Y = Y(u(t,n)) = (Y_1, Y_2, Y_3)^T$, we have

$$\begin{aligned} A_n &= -q_n B_n + E_{n-1} & D_n &= p_n B_{n+1} + r_n C_{n+1} - \lambda B_{n+1} \\ F_n &= B_{n+1} & G_n &= r_{n-1} B_{n-1} - q_{n-1} r_{n-1} C_{n-1} \\ H_n &= r_{n-1} C_{n-1} & I_n &= -q_n B_{n+1} - p_n C_n + E_{n+1} + \lambda C_n \end{aligned}$$

and

$$X = \lambda Y + (L_1 - \lambda L_2) \begin{pmatrix} B_n \\ C_n \\ E_n \end{pmatrix}$$

in which

$$L_1 = \begin{pmatrix} E p_n E - p_n & E r_n E - E^{-1} r_n & q_n(E-1) \\ -E q_n E + q_{n-1} & (1-E) p_n & E^2 - E^{-1} \\ r_n E^2 - E^{-1} r_n - p_n(E-1) q_{n-1} & E^{-1} q_n r_n - q_n r_n E & p_n \Delta \end{pmatrix}$$

$$L_2 = \begin{pmatrix} \Delta E & 0 & 0 \\ 0 & 1 - E & 0 \\ (1-E) q_n & 0 & \Delta \end{pmatrix}.$$

Then, just like the treatment of the Toda lattice, we can find that the B–M lattice satisfies the conditions of theorem 1. The first equation is

$$u_{nt} = K^{(0)} = \begin{pmatrix} r_{n+1} - r_{n-1} \\ p_n - p_{n+1} \\ q_{n-1}r_{n-1} - q_n r_n \end{pmatrix} \tag{51}$$

and the related time matrix is

$$N^{(0)} = \begin{pmatrix} 0 & 0 & 1 \\ r_n & 0 & 0 \\ -q_{n-1}r_{n-1} & r_{n-1} & \lambda - p_n \end{pmatrix}.$$

The isospectral hierarchy is

$$u_{nt} = K^{(l)}(u(t,n)) = L^l K^{(0)}(u(t,n)) \qquad l = 0, 1, \ldots \tag{52}$$

where the recursion operator, strong and hereditary symmetry for equation (52), $L = L_1 L_2^{-1} = (L_{ij})_{3\times 3}$ reads as

$$\begin{aligned} &L_{11} = (Ep_n - p_n E^{-1} + q_n \Delta_- \Delta_+^{-1} q_n)\Delta^{-1} && L_{12} = (E^{-1}r_n - Er_n E)\Delta_-^{-1} \\ &L_{13} = q_n(1 + E^{-1})^{-1} && L_{21} = -\Delta_- \Delta_+^{-1} q_n \Delta^{-1} \\ &L_{22} = \Delta_- p_n \Delta_-^{-1} && L_{23} = (E^2 - E^{-1})\Delta^{-1} \\ &L_{31} = (r_n E - E^{-1} r_n E^{-1})\Delta^{-1} && L_{32} = (q_n r_n E - E^{-1} q_n r_n)\Delta_-^{-1} \\ &L_{33} = p_n \end{aligned} \tag{53}$$

where $\Delta_- = E - 1$ and $\Delta_+ = E + 1$.

Next, L can be written in the following form

$$L = \theta J \tag{54}$$

in which θ and $J = (J_{ij})_{3\times 3}$ are both skew-symmetric, described respectively by

$$\theta = \begin{pmatrix} \Delta & 0 & 0 \\ 0 & 0 & 1 - E \\ 0 & E^{-1} - 1 & 0 \end{pmatrix} \tag{55}$$

and

$$\begin{aligned} &J_{11} = \Delta^{-1}[Ep_n - p_n E^{-1} + q_n(1 + E^{-1})^{-1}\Delta\Delta_+^{-1} q_n]\Delta^{-1} \\ &J_{12} = \Delta^{-1}(E^{-1}r_n - Er_n E)\Delta_-^{-1} \qquad J_{13} = \Delta^{-1} q_n (1 + E^{-1})^{-1} \\ &J_{21} = (1 - E^{-1})^{-1}(E^{-1} r_n E^{-1} - r_n E)\Delta^{-1} \\ &J_{22} = (1 - E^{-1})^{-1}(E^{-1} q_n r_n - q_n r_n E)\Delta_-^{-1} \\ &J_{23} = -(1 - E^{-1})^{-1} p_n \qquad J_{31} = \Delta_+{}^{-1} q_n \Delta^{-1} \\ &J_{32} = -p_n \Delta_-^{-1} \qquad J_{33} = \Delta^{-1}(E^2 - E^{-2} + \Delta)\Delta^{-1}. \end{aligned} \tag{56}$$

Now we rewrite the first equation (51) as

$$u_{nt} = \theta f^{(0)}(u(t,n)) \qquad f^{(0)} = (r_n, q_n r_n, p_n)^T$$

and it is easy to verify that $f^{(0)\prime} = f^{(0)\prime *}$. So, according to theorem 2, equation (54) is an implectic-symplectic factorization, the hierarchy (52) is a complete Liouville integrable and the Hamiltonian structures are

$$u_{nt} = \theta \frac{\delta H^{(l)}}{\delta u} = \theta L^* \frac{\delta H^{(l-1)}}{\delta u} = \cdots = \theta L^{*l} \frac{\delta H^{(0)}}{\delta u} = \theta L^{*l} f^{(0)}(u(t,n)) \qquad l = 0, 1, \ldots \tag{57}$$

where the conserved quantities $H^{(l)}$ are obtained from equation (29). The first two of these are

$$H^{(0)}(u(t,n)) = \sum_{n=-\infty}^{+\infty}\left(q_n r_n + \frac{1}{2}p_n^2\right)$$
$$H^{(1)}(u(t,n)) = \sum_{n=-\infty}^{+\infty}\left[\frac{1}{3}p_n^3 + q_n r_n(p_n + p_{n+1}) - r_n r_{n-1} + 1\right].$$

5.3. Ablowitz–Ladik lattice

The linear problem pair in which we are interested, related to the Ablowitz–Ladik (A–L) lattice [5, 22–25], reads

$$E\phi = M\phi \qquad M = \begin{pmatrix}\lambda & Q_n\\ R_n & \frac{1}{\lambda}\end{pmatrix} \qquad u_n = \begin{pmatrix}Q_n\\ R_n\end{pmatrix} \qquad \phi = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$$
$$\phi_t = N\phi \qquad N = \begin{pmatrix}A_n & B_n\\ C_n & D_n\end{pmatrix}. \tag{58}$$

Now, we consider the matrix equation

$$M'[X - \lambda^\alpha Y] = (EN)M - MN \tag{59}$$

where $X = X(u(t,n)) = (X_1, X_2)^T$ and $Y = Y(u(t,n)) = (Y_1, Y_2)^T$. This leads to

$$A_n = \frac{1}{\lambda}(E-1)^{-1}(-R_n E B_n + Q_n C_n) + a_0$$
$$D_n = \lambda(E-1)^{-1}(R_n B_n - Q_n E C_n) + d_0 \tag{60}$$

and

$$X = \lambda^\alpha Y + \left(\lambda L_1 - \frac{1}{\lambda}L_2\right)\begin{pmatrix}B_n\\ C_n\end{pmatrix} + (a_0 - d_0)\begin{pmatrix}Q_n\\ -R_n\end{pmatrix} \tag{61}$$

where $a_0 = A_n|_{u=0}$ and $d_0 = D_n|_{u=0}$ are all constants independent of u, and

$$L_1 = \begin{pmatrix}-1 & 0\\ 0 & E\end{pmatrix} + \begin{pmatrix}-Q_n\\ R_n E\end{pmatrix}(E-1)^{-1}(R_n, -Q_n E) \tag{62}$$

$$L_2 = \begin{pmatrix}-E & 0\\ 0 & 1\end{pmatrix} - \begin{pmatrix}-Q_n E\\ R_n\end{pmatrix}(E-1)^{-1}(R_n E, -Q_n). \tag{63}$$

Moreover, we present the inverse operators of L_1 and L_2

$$L_1^{-1} = \begin{pmatrix}-1 & 0\\ 0 & E^{-1}\end{pmatrix} + \begin{pmatrix}Q_n\\ R_{n-1}\end{pmatrix}(E-1)^{-1}(R_n, Q_n)\frac{1}{\mu_n} \tag{64}$$

$$L_2^{-1} = \begin{pmatrix}-E^{-1} & 0\\ 0 & 1\end{pmatrix} - \begin{pmatrix}Q_{n-1}\\ R_n\end{pmatrix}(E-1)^{-1}(R_n, Q_n)\frac{1}{\mu_n} \tag{65}$$

in which $\mu_n = 1 - Q_n R_n$.

If we set $Y = 0$, $a_0 = -d_0 = \frac{1}{2}\lambda^{-2}$ and $(B_n, C_n)^T = -\lambda^{-1}(Q_{n-1}, R_n)^T$ in equation (61), we can find the non-zero solutions in $\mathcal{V}_2$ and $\mathcal{Q}_2^{(0-)}(\lambda)$

$$X = \hat{K}^{(0)}(u(t,n)) = \mu_n\begin{pmatrix}Q_{n-1}\\ -R_{n+1}\end{pmatrix}$$
$$N = \hat{N}^{(0)}(\lambda, u(t,n)) = \begin{pmatrix}\frac{1}{2}\lambda^{-2} & -Q_{n-1}\lambda^{-1}\\ -R_n\lambda^{-1} & -\frac{1}{2}\lambda^{-2} + R_n Q_{n-1}\end{pmatrix} \tag{66}$$

satisfying

$$M'[X] = (EN)M - MN. \tag{67}$$

Next, in the case of $Y \neq 0$, taking $\alpha = -2, a_0 = d_0 = 0$ and $(B_n, C_n)^T = \lambda^{-1}(b_0, c_0)^T$ (b_0 and c_0 are both independent of λ) in equation (61) we find

$$\begin{pmatrix} B_n \\ C_n \end{pmatrix} = L_2^{-1} Y \qquad X = \hat{L} Y$$

where the recursion operator $\hat{L} = L_1 L_2^{-1}$ is described by

$$\begin{aligned} \hat{L} = \begin{pmatrix} E^{-1} & 0 \\ 0 & E \end{pmatrix} - \begin{pmatrix} -Q_n \\ R_n E \end{pmatrix} (E-1)^{-1} (R_n E^{-1}, Q_n E) \\ + \mu_n \begin{pmatrix} Q_{n-1} \\ -E R_n \end{pmatrix} (E-1)^{-1} (R_n, Q_n) \frac{1}{\mu_n}. \end{aligned} \tag{68}$$

It is easy to check $N|_{u=0} = 0$ and $X|_{u=0} = 0$ if $Y \in \mathcal{V}_2$. In addition, similar to the discussion of the Toda lattice, one can find that equation (67) only admits zero solutions in $\mathcal{V}_2$ and $\mathcal{Q}_2^{(0-)}(\lambda)$ as $N|_{u=0} = 0$.

So, in the light of theorem 1, we have the isospectral evolution equations

$$u_{nt} = \hat{K}^{(l)}(u(t,n)) = \hat{L}^l \hat{K}^{(0)}(u(t,n)) \qquad l = 0, 1, \ldots \tag{69}$$

and the recursion operator $\hat{L}$ is a strong and hereditary symmetry for the every member of equation (69).

On the other hand, setting $\alpha = 2$ in equation (61), one can find that the linear problem pair (58) also satisfies the conditions of theorem 1 on the spaces $\mathcal{V}_2$ and $\mathcal{Q}_2^{(0+)}(\lambda)$. In this case, another hierarchy is obtained

$$u_{nt} = \widetilde{K}^{(l)}(u(t,n)) = \widetilde{L}^l \widetilde{K}^{(0)}(u(t,n)) = \widetilde{L}^l \mu_n \begin{pmatrix} Q_n \\ -R_{n-1} \end{pmatrix} \qquad l = 0, 1, \ldots \tag{70}$$

where the recursion operator $\widetilde{L}$, also a strong and hereditary symmetry for equation (70), is defined by

$$\begin{aligned} \widetilde{L} = L_2 L_1^{-1} = \begin{pmatrix} E & 0 \\ 0 & E^{-1} \end{pmatrix} + \begin{pmatrix} -Q_n E \\ R_n \end{pmatrix} (E-1)^{-1} (R_n E, Q_n E^{-1}) \\ + \mu_n \begin{pmatrix} -E Q_n \\ R_{n-1} \end{pmatrix} (E-1)^{-1} (R_n, Q_n) \frac{1}{\mu_n}. \end{aligned} \tag{71}$$

Taking

$$K^{(0)}(u(t,n)) = \begin{pmatrix} Q_n \\ -R_n \end{pmatrix} \tag{72}$$

and noting that $\hat{L}^{-1} = \widetilde{L}$ and

$$\hat{L} \widetilde{K}^{(0)} = \widetilde{L} \hat{K}^{(0)} = K^{(0)}$$

we can combine equations (69) and (70):

$$u_{nt} = K^{(l)}(u(t,n)) = L^l K^{(0)}(u(t,n)) \qquad L = \hat{L}, l \in Z. \tag{73}$$

Now, we factorize L into

$$L = \theta J \tag{74}$$

where

$$\theta = \mu_n \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and

$$J = \frac{1}{\mu_n}\begin{pmatrix} 0 & E \\ E^{-1} & 0 \end{pmatrix} + \frac{1}{\mu_n}\begin{pmatrix} R_n E \\ Q_n \end{pmatrix}(E-1)^{-1}(R_n E^{-1}, Q_n E) + \begin{pmatrix} E R_n \\ Q_{n-1} \end{pmatrix}(E-1)^{-1}(R_n, Q_n)\frac{1}{\mu_n}$$

are both skew-symmetric. With this in mind, we can rewrite the hierarchy (73) as

$$u_{nt} = \theta f^{(l)}(u(t,n)) = \theta L^{*l} f^{(0)}(u(t,n)) \qquad f^{(0)} = \frac{1}{\mu_n}\begin{pmatrix} R_{n+1} \\ Q_{n-1} \end{pmatrix} \qquad l \in Z. \tag{75}$$

Obviously, every equation in the hierarchy (73) can act as the role of 'the first equation'. But here, for convenience, $u_{nt} = K^{(1)} = \hat{K}^{(0)} = \theta f^{(1)}$ may be more suitable because it is easy to find from equation (66) that $f^{(1)} = (R_{n+1}, Q_{n-1})^T$ and to verify $f^{(1)\prime} = f^{(1)\prime *}$. So, equation (74) is an implectic-symplectic factorization, every equation in the hierarchy (73) is completely integrable in the Liouville sense and the Hamiltonian structure is

$$u_{nt} = \theta \frac{\delta H^{(l)}}{\delta u} = \theta L^* \frac{\delta H^{(l-1)}}{\delta u} = \cdots = \theta L^{*l} \frac{\delta H^{(0)}}{\delta u} = \theta L^{*l} f^{(0)}(u(t,n)) \qquad l \in Z. \tag{76}$$

According to formula (29), we write three of the conserved quantities $H^{(l)}$:

$$H^{(-1)}(u(t,n)) = \frac{1}{2}\sum_{n=-\infty}^{+\infty}(Q_n R_{n-1} + Q_{n+1} R_n)$$

$$H^{(0)}(u(t,n)) = -\sum_{n=-\infty}^{+\infty}\ln(1 - Q_n R_n)$$

$$H^{(1)}(u(t,n)) = \frac{1}{2}\sum_{n=-\infty}^{+\infty}(Q_{n-1} R_n + Q_n R_{n+1}).$$

5.4. A new lattice

From the following linear problem pair

$$E\phi = M\phi \qquad M = \begin{pmatrix} 0 & 1 & 0 \\ v_n & p_n + \lambda & 1 \\ w_n & 0 & 0 \end{pmatrix} \qquad u_n = \begin{pmatrix} p_n \\ v_n \\ \ln w_n \end{pmatrix} \qquad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$$
$$\phi_t = N\phi \qquad N = \begin{pmatrix} A_n & B_n & C_n \\ D_n & E_n & F_n \\ G_n & H_n & I_n \end{pmatrix} \tag{77}$$

we can derive a new lattice which we call the B–M(II) lattice [20]. Along the lines of the previous discussion, it is not difficult to find that equation (77) satisfies the conditions of theorems 1 and 2. Here, we only list the main results.

The matrix equation

$$M'[X - \lambda Y] = (EN)M - MN \tag{78}$$

gives

$$A_n = \lambda B_n - p_{n-1}B_n + E_{n-1} \qquad D_n = v_n B_{n+1} + w_n C_{n+1}$$
$$F_n = B_{n+1} \qquad G_n = -\lambda w_{n-1}C_{n-1} + w_{n-1}B_{n-1} - p_{n-1}w_{n-1}C_{n-1}$$
$$H_n = w_{n-1}C_{n-1} \qquad I_n = -\lambda B_{n+1} - p_n B_{n+1} - v_n C_n + E_{n+1}$$

and

$$X = \lambda Y + (L_1 - \lambda L_2)\begin{pmatrix} B_n \\ C_n \\ E_n \end{pmatrix} \tag{79}$$

where L_1 and L_2 are described as

$$L_1 = \begin{pmatrix} E v_n E - v_n & E w_n E - E^{-1} w_n & p_n(E-1) \\ v_n(p_{n-1} - p_n E) + w_n E^2 - E^{-1} w_n & E^{-1} p_n w_n - p_n w_n E & v_n \Delta \\ p_{n-1} - E p_n E & (1-E) v_n & E^2 - E^{-1} \end{pmatrix} \tag{80}$$

$$L_2 = \begin{pmatrix} 0 & 0 & 1-E \\ v_n(E-1) & w_n E - E^{-1} w_n & 0 \\ \Delta E & 0 & 0 \end{pmatrix}. \tag{81}$$

The isospectral hierarchy is

$$u_{nt} = K^{(l)}(u) = L^l K^{(0)}(u) \qquad l = 0, 1, \ldots \tag{82}$$

in which the recursion operator $L = L_1 L_2^{-1} = (L_{ij})_{3\times 3}$ and

$$L_{11} = -p_n \qquad L_{12} = (E w_n E - E^{-1} w_n)(w_n E - E^{-1} w_n)^{-1}$$
$$L_{13} = (E v_n - v_n E^{-1})\Delta^{-1} - (E w_n E - E^{-1} w_n)(w_n E - E^{-1} w_n)^{-1} v_n (E+1)^{-1}$$
$$L_{21} = -v_n(1 + E^{-1}) \qquad L_{22} = (E^{-1} p_n w_n - p_n w_n E)(w_n E - E^{-1} w_n)^{-1}$$
$$L_{23} = [v_n(E^{-1} - 1)p_n + w_n E - E^{-1} w_n E^{-1}]\Delta^{-1}$$
$$\qquad - (E^{-1} p_n w_n - p_n w_n E)(w_n E - E^{-1} w_n)^{-1} v_n (E+1)^{-1}$$
$$L_{31} = -(E + 1 + E^{-1}) \qquad L_{32} = (1-E) v_n (w_n E - E^{-1} w_n)^{-1}$$
$$L_{33} = -\Delta p_n \Delta^{-1} - (1-E) v_n (w_n E - E^{-1} w_n)^{-1} v_n (E+1)^{-1}.$$

The first equation in (82) is

$$u_{nt} = K^{(0)} = \begin{pmatrix} v_{n+1} - v_n \\ v_n(p_{n-1} - p_n) + w_n - w_{n-1} \\ p_{n-1} - p_{n+1} \end{pmatrix}. \tag{83}$$

We can decompose L into

$$L = \theta J \tag{84}$$

where

$$\theta = \begin{pmatrix} 0 & (1-E) v_n & -\Delta \\ v_n(E^{-1} - 1) & E^{-1} w_n - w_n E & 0 \\ -\Delta & 0 & 0 \end{pmatrix}$$

$J = (J_{ij})_{3\times 3}$ and

$$J_{11} = \Delta^{-1}(E + 1 + E^{-1}) \qquad J_{12} = (E^{-1} + 1)^{-1} v_n (w_n E - E^{-1} w_n)^{-1}$$
$$J_{13} = p_n \Delta^{-1} - (E^{-1} + 1)^{-1} v_n (w_n E - E^{-1} w_n)^{-1} v_n (E+1)^{-1}$$
$$J_{21} = (w_n E - E^{-1} w_n)^{-1} v_n (E+1)^{-1}$$

$$
\begin{aligned}
J_{22} = &-(w_nE - E^{-1}w_n)^{-1}(E^{-1}p_nw_n - p_nw_nE)(w_nE - E^{-1}w_n)^{-1}\\
&+ (w_nE - E^{-1}w_n)^{-1}v_n(E^{-1} - 1)\Delta^{-1}(E - 1)v_n(w_nE - E^{-1}w_n)^{-1}\\
J_{23} = &(w_nE - E^{-1}w_n)^{-1}(w_nE - E^{-1}w_nE^{-1})\Delta^{-1}\\
&+ (w_nE - E^{-1}w_n)^{-1}(E^{-1}p_nw_n - p_nw_nE)(w_nE - E^{-1}w_n)^{-1}v_n(E + 1)^{-1}\\
&- (w_nE - E^{-1}w_n)^{-1}v_n(E^{-1} - 1)\\
&\times \Delta^{-1}(E - 1)v_n(w_nE - E^{-1}w_n)^{-1}v_n(E + 1)^{-1}\\
J_{31} = &\Delta^{-1}p_n - (E^{-1} + 1)^{-1}v_n(w_nE - E^{-1}w_n)^{-1}v_n(E + 1)^{-1}\\
J_{32} = &\Delta^{-1}(E^{-1}w_n - Ew_nE)(w_nE - E^{-1}w_n)^{-1}\\
&+ (E^{-1} + 1)^{-1}v_n(w_nE - E^{-1}w_n)^{-1}(E^{-1}p_nw_n - p_nw_nE)(w_nE - E^{-1}w_n)^{-1}\\
&- (E^{-1} + 1)^{-1}v_n(w_nE - E^{-1}w_n)^{-1}v_n(E^{-1} - 1)\\
&\times \Delta^{-1}(E - 1)v_n(w_nE - E^{-1}w_n)^{-1}\\
J_{33} = &-\Delta^{-1}(Ev_n - v_nE^{-1})\Delta^{-1}\\
&+ \Delta^{-1}(Ew_nE - E^{-1}w_n)(w_nE - E^{-1}w_n)^{-1}v_n(E + 1)^{-1}\\
&+ (E^{-1} + 1)^{-1}v_n(w_nE - E^{-1}w_n)^{-1}(w_nE - E^{-1}w_nE^{-1})\Delta^{-1}\\
&+ (E^{-1} + 1)^{-1}v_n(w_nE - E^{-1}w_n)^{-1}(E^{-1}p_nw_n - p_nw_nE)\\
&\times (w_nE - E^{-1}w_n)^{-1}v_n(E + 1)^{-1}\\
&+ (E^{-1} + 1)^{-1}v_n(w_nE - E^{-1}w_n)^{-1}v_n(E^{-1} - 1)\\
&\times \Delta^{-1}(E - 1)v_n(w_nE - E^{-1}w_n)^{-1}v_n(E + 1)^{-1}.
\end{aligned}
$$

Equation (83) can be written in the form

$$u_{nt} = \theta f^{(0)}(u) \qquad f^{(0)} = (p_n, -1, 0)^T \qquad f^{(0)\prime} = f^{(0)\prime *}.$$

So, equation (84) is an implectic-symplectic factorization, every equation in the hierarchy (82) is completely integrable in the Liouville sense and the Hamiltonian structure is

$$u_{nt} = \theta\frac{\delta H^{(l)}}{\delta u} = \theta L^*\frac{\delta H^{(l-1)}}{\delta u} = \cdots = \theta L^{*l}\frac{\delta H^{(0)}}{\delta u} = \theta L^{*l} f^{(0)}(u) \qquad l = 0, 1, \ldots. \tag{85}$$

It is not difficult to obtain the first conserved quantity

$$H^{(0)}(u) = \sum_{n=-\infty}^{+\infty}\left(\frac{1}{2}p_n^2 - v_n\right).$$

6. Conclusions

We have described a method for constructing the Hamiltonian structures of isospectral evolution equations using the related recursion operator. This method allows us, in the same procedure, to obtain, directly from the discrete Lax pair, the isospectral hierarchy, the recursion operator (which is a strong and hereditary symmetry of the flows), the (multi-)Hamiltonian structures and the Liouville complete integrability of the evolution equations. Furthermore, this method can certainly be applied to continuous soliton systems.

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